Asymptotics of Some Convolutional Recurrences

Asymptotics of Some Convolutional Recurrences

Edward A. Bender *

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112

ebender@ucsd.edu

Adri B. Olde Daalhuis

Maxwell Institute and School of Mathematics
The University of Edinburgh
Edinburgh, EH9 3JZ, UK

A.OldeDaalhuis@ed.ac.uk

Zhicheng Gao †

School of Mathematics and Statistics
Carleton University
Ottawa, Ontario K1S5B6

zgao@math.carleton.ca

L. Bruce Richmond* and Nicholas Wormald‡

Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario N2L3G1

lbrichmond@uwaterloo.ca, nwormald@uwaterloo.ca

Submitted: Apr 7, 2009; Accepted: Dec 14, 2009; Published: Jan 5, 2010

Abstract

We study the asymptotic behavior of the terms in sequences satisfying recurrences of the form $a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n,k) a_k a_{n-k}$ where, very roughly speaking, $f(n,k)$ behaves like a product of reciprocals of binomial coefficients. Some examples of such sequences from map enumerations, Airy constants, and Painlevé I equations are discussed in detail.

1 Main results

There are many examples in the literature of sequences defined recursively using a convolution. It often seems difficult to determine the asymptotic behavior of such sequences. In this note we study the asymptotics of a general class of such sequences. We prove

*Research supported by NSERC
†Research supported by NSERC
‡Research supported by NSERC and Canada Research Chair Program

subexponential growth by using an iterative method that may be useful for other recurrences. By subexponential growth we mean that, for every constant $D > 1$, $a_n = o(D^n)$ as $n \to \infty$. Thus our motivation for this note is both the method and the applications we give.

Let $d > 0$ be a fixed integer and let $f(n,k) \geqslant 0$ be a function that behaves like a product of some powers of reciprocals of binomial coefficients, in a general sense to be specified in Theorem 1. We deal with the sequence a_n for $n \geqslant d$ where $a_d, a_{d+1}, \cdots, a_{2d-1} \geqslant 0$ are arbitrary and, when $n \geqslant 2d$,

$$a_n \;=\; a_{n-1} + \sum_{k=d}^{n-d} f(n,k) a_k a_{n-k}. \tag{1}$$

Without loss of generality,

$$\text{we assume that } f(n,k) = f(n,n-k)$$

since we can replace $f(n,k)$ and $f(n,n-k)$ in (1) with $\frac{1}{2}(f(n,k) + f(n,n-k))$.

Theorem 1 proves subexponential growth. Theorem 2 provide more accurate estimates under additional assumptions. In Section 2, we apply the corollary to some examples.

Theorem 1 (Subexponential growth) *Let a_n be defined by recursion (1) with $a_d > 0$. Suppose there is a function $R(x)$ defined on $(0, 1/2]$, an $\alpha > 0$ and an r such that*

(a) $0 < R(x) < r < 1$,

(b) $\lim_{x \to 0+} R(x) = 0$,

(c) $0 \leqslant f(n,k) = O\left(n^{-\alpha} R^{k-d}(k/n)\right)$ *uniformly for* $d \leqslant k \leqslant n/2$.

Then a_n grows subexponentially; in fact,

$$a_n \;=\; \left(1 + O(n^{-\alpha})\right) a_{n-1}. \tag{2}$$

Proof: We first note that the a_n are non-decreasing when $n \geqslant 2d - 1$.

Our proof is in three steps. We first prove that $a_n = O(C^n)$ for some constant $C > 2$. We then prove that C can be chosen very close to 1. Finally we deduce (2) and subexponential growth.

First Step: Since the bound in (c) is bounded by some constant times the geometric series $n^{-\alpha} r^{k-d}$ with ratio less than 1, $\sum_{k=d}^{n-d} f(n,k) = O(n^{-\alpha})$. Hence we can choose M so large that $\sum_{k=d}^{n-d} f(n,k) < 1/4$ when $n > M$. Next choose $C \geqslant 2$ so large ($C = \max\{a_d, a_{d+1}, ..., a_{2d-1}, a_M, 2\}$ will do) that $a_n < 2C^n$ for $n \leqslant M$. By induction, using the recursion (1), we have for $n > M$

$$a_n \;<\; 2C^{n-1} + (1/4)4C^n \;\leqslant\; C^n + C^n \;=\; 2C^n.$$

Second Step: By (b) there is a λ in $(0, 1/2)$ such that $R(x) < \frac{1}{2C}$ for $0 < x < \lambda$. Fix any $D \leqslant C$ such that $a_n = O(D^n)$, which is true for $D = C$ by the First Step.

Split the sum in (1) into $\lambda n \leqslant k \leqslant (1-\lambda)n$ and the rest, calling the first range of k the "center" and the rest the "tail". Noting $r < 1$, the center sum is bounded by

$$2 \sum_{k=\lambda n+1}^{n/2} f(n,k) a_k a_{n-k} \;=\; O\left(D^n \sum_{k=\lambda n+1}^{n/2} r^{k-d} \right) \;=\; O\left((r^\lambda D)^n \right). \tag{3}$$

Since a_j are increasing, the tail sum is bounded by

$$\begin{aligned} 2 \sum_{k=d}^{\lambda n} f(n,k) a_k a_{n-k} \;&=\; O(n^{-\alpha}) a_{n-1} \sum_{k=d}^{\lambda n} R(x)^{k-d} D^k \qquad (4) \\ &=\; O(n^{-\alpha}) a_{n-1} \sum_{k=d}^{\lambda n} (DR(x))^{k-d} \;=\; O\left(n^{-\alpha} a_{n-1} \right), \end{aligned}$$

where the last equality follows from the fact that $DR(x) < 1/2$. Combining (3) and (4),

$$a_n \;=\; \left(1 + O(n^{-\alpha})\right) a_{n-1} + O((r^\lambda D)^n). \tag{5}$$

When $r^\lambda D > 1$, induction on n easily leads to $a_n = O((r^\lambda D')^n)$ for any $D' > D$, an exponential growth rate no larger than $r^\lambda D'$.

Since r^λ has a fixed value less than one, we can iterate this process, replacing D by $r^\lambda D'$ at the start of the Second Step. We finally obtain a growth rate $D > 1$ with $r^\lambda D < 1$. This completes the second step.

Third Step: With the value of D just obtained, the last term in (5) is exponentially small and hence is $O(n^{-\alpha} a_{n-1})$. Thus we obtain (2) which immediately implies subexponential growth of a_n, since $1 + O(n^{-\alpha}) < D$ for any $D > 1$ and sufficiently large n. ∎

To say more than (2), we need additional information about the behavior of the $f(n,k)$. When $f(n,k)/f(n,d)$ is small for each k in the range $d+1 \leqslant k \leqslant n-d-1$, the first and last terms dominate the sum. The following theorem is based on this observation.

Theorem 2 (Asymptotic behavior) *Assume (a)–(c) of Theorem 1 hold. Suppose further that there is a $\beta > 0$ such that*

$$\frac{f(n,k)}{f(n,d)} \;=\; O(n^{-\beta} r^{k-d-1}) \quad \textit{uniformly for} \quad d+1 \leqslant k \leqslant n/2. \tag{6}$$

Then

$$\log a_n \;=\; 2a_d \sum_{k=2d+1}^{n} f(k,d) + O\left(\sum_{k=2d}^{n} f(k,d) \left(k^{-\alpha} + k^{-\beta} \right) \right). \tag{7}$$

Proof: We assume $n > 2d$. Remove the $k = d$ and $k = n - d$ terms from the sum in (1). We first deal with the remaining sum. Theorem 1 gives $a_k = O(D^k)$ for all $D > 1$, so we can assume $D < 1/r$. Using (6)

$$\begin{aligned}\sum_{k=d+1}^{n-d-1} f(n,k) a_k a_{n-k} &= O\Big(f(n,d) n^{-\beta} a_{n-1}\Big) \sum_{k=d+1}^{n/2} r^{k-d-1} D^k \\ &= O\Big(f(n,d) n^{-\beta} a_{n-1}\Big).\end{aligned}$$

Combining this with (1), we obtain

$$\begin{aligned}a_n &= a_{n-1} + 2a_d f(n,d) a_{n-d} + f(n,d) O(n^{-\beta}) a_{n-1} \\ &= a_{n-1}\Big(1 + 2a_d f(n,d) + \{O(n^{-\alpha}) + O(n^{-\beta})\} f(n,d)\Big),\end{aligned}$$

Taking logarithms and noting for expansion purposes that $f(n,d) = O(n^{-\alpha})$, we obtain

$$\log a_n - \log a_{n-1} = 2a_d f(n,d) + O\left(\left(n^{-\alpha} + n^{-\beta}\right) f(n,d)\right).$$

Sum over n starting with $n = 2d + 1$. The theorem follows immediately when we note that the constant terms can be incorporated into the $O(\ )$ in (7) since the sum therein is bounded below by a nonzero constant. ∎

Corollary 1 *Assume the conditions of Theorem 2 hold and $f(n,d) = \Theta(n^{-\alpha})$.*

- *If $\alpha < 1$, then $a_n = \exp\left(\Theta\left(n^{1-\alpha}\right)\right)$.*
- *If $\alpha > 1$, then $a_n = K + O(n^{1-\alpha})$ for some constant K.*
- *If $f(n,d) - A/n$ are the terms of a convergent series, then $a_n \sim Cn^{2Aa_d}$ for some positive constant C.*

Proof: Since $\alpha > 0$ and $\beta > 0$, (7) gives $\log a_n = \Theta(\sum_{k=2d+1}^{n} k^{-\alpha})$. The case $\alpha < 1$ follows immediately; for $\alpha > 1$, we see that a_n is bounded and nondecreasing and therefore has a limit K. For $m > n$, (2) gives $\log(a_m/a_n) = O(n^{1-\alpha})$ uniformly in m. Letting $m \to \infty$, we obtain the claim regarding $\alpha > 1$.

For $\alpha = 1$, the first sum in (7) is $A \log n + B + o(1)$ for some constant B, and the last sum in (7) converges. ∎

2 Examples

We apply Theorem 2 and Corollary 1 to some recursions which arise from combinatorial applications. In our examples, $f(n,k)$ behaves like a product of the reciprocal of binomial coefficients, which satisfies the conditions of Theorems 1 and 2. A more general case of interest is when $f(n,k)$ takes the form of the product of functions like

$$g(n,k) = \frac{[a]_k\ [a]_{n-k}}{[a]_n}$$

for some constant $a > 0$, where $[x]_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(k)}$, the rising factorial. We note that when $a = 1$, $g(n,k) = \binom{n}{k}^{-1}$.

We begin with some useful bounds. When $a > 0$ and $1 \leqslant k \leqslant n/2$,

$$\begin{aligned} g(n,k) &= \prod_{j=0}^{k-1} \frac{a+j}{a+n-k+j} \;<\; \left(\frac{a+k}{a+n}\right)^k \\ &\leqslant (k/n)^k \left(\frac{1+a/k}{1+a/n}\right)^k \;=\; O\left((k/n)^k\right) \;=\; O\left(n^{-1}(3k/2n)^{k-1}\right) \end{aligned} \tag{8}$$

since $k(2/3)^{k-1}$ is bounded. So g satisfies the condition on f in Theorem 1(c), with $\alpha = 1$. Similarly, when $a > 0$ and $d \leqslant k \leqslant n/2$,

$$\frac{g(n,k)}{g(n,d)} \;=\; \prod_{j=0}^{k-d-1} \frac{a+d+j}{a+n-k+d+j} \;=\; O\left(n^{-1}(3k/2n)^{k-d-1}\right). \tag{9}$$

This is in accordance with (6) with $\beta = 1$.

Example 1 (Map enumeration constants) There are numbers t_n appearing in the asymptotic enumeration of maps in an orientable surface of genus n, whose value does not concern us here. Define u_n by

$$t_n \;=\; 8\frac{[1/5]_n\,[4/5]_{n-1}}{\Gamma\left(\frac{5n-1}{2}\right)} \left(\frac{25}{96}\right)^n u_n.$$

Then $u_1 = 1/10$ and u_n satisfies the following recursion [3]

$$u_n \;=\; u_{n-1} + \sum_{k=1}^{n-1} f(n,k) u_k u_{n-k} \quad \text{for} \quad n \geqslant 2, \tag{10}$$

where

$$f(n,k) \;=\; \frac{[1/5]_k\,[1/5]_{n-k}}{[1/5]_n}\,\frac{[4/5]_{k-1}\,[4/5]_{n-k-1}}{[4/5]_{n-1}}.$$

From the observations above, the conditions of Theorem 2 are satisfied with $d = 1$, $R(\lambda) = (3\lambda/2)^2$ and $\alpha = \beta = 2$. Hence, $u_n \sim K$ for some constant K. Unlike the proof in [3], this does not depend on the value of u_1. ∎

Example 2 (Airy constants) The *Airy constants* Ω_n are determined by $\Omega_1 = 1/2$ and the recurrence [7]

$$\Omega_n \;=\; (3n-4)n\Omega_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} \Omega_k \Omega_{n-k} \quad \text{for} \quad n \geqslant 2.$$

Let $\Omega_n = n!\,[2/3]_{n-1}\,3^n a_n$. Then a_n satisfies (1) with $d = 1$ and

$$f(n,k) = \frac{[2/3]_{k-1}\,[2/3]_{n-k-1}}{[2/3]_{n-1}}.$$

Theorem 2 applies with $d = 1$, $R(\lambda) = 3\lambda/2$ and $\alpha = \beta = 1$. Since

$$f(n,1) \;=\; \frac{1}{n-4/3} \;=\; \frac{1}{n} + \frac{4/3}{n(n-4/3)}$$

and $a_1 = 1/6$, we have $a_n \sim Cn^{1/3}$ for some constant C.

We note that it is possible to apply the result of Olde Daalhuis [13] to obtain a full asymptotic expansion for Ω_n. Let

$$A_n = \frac{\Omega_n}{3^n n!}.$$

Then the recursion for Ω_n becomes

$$A_n = (n-4/3)\,A_{n-1} + \sum_{k=1}^{n-1} A_k A_{n-k},\; n \geqslant 2.$$

It follows that the formal series

$$F(z) = \sum_{n\geqslant 1} \frac{A_n}{z^n}$$

satisfies the Riccati equation

$$F^{'}(z) + \left(1 + \frac{1}{3z}\right) F(z) - F^2(z) - \frac{1}{6z} = 0.$$

It then follows from the result of Olde Daalhuis [13] that

$$A_n \sim \frac{1}{2\pi} \sum_{k=0}^{\infty} b_k \Gamma(n-k),\; \text{as } n \to \infty,$$

where $b_0 = 1$ and b_k can be computed using the recursion

$$b_k = \frac{-2}{k} \sum_{j=2}^{k+1} b_{k+1-j} A_j, \quad k \geqslant 1.$$

In particular, we have

$$\Omega_n \sim \frac{1}{2\pi}\Gamma(n) 3^n n! = \frac{1}{2\pi n}(n!)^2 3^n, \quad \text{as } n \to \infty.$$

It is well known that solutions to the Riccati equation have infinitely many singularities, hence $F(z)$ (via its Borel transform [2]) cannot satisfy a linear ODE with polynomial coefficients. This implies that the sequence A_n (and hence Ω_n) is not holonomic. ∎

Example 3 The following recursion, with $\ell > 0$ and $\ell \neq 1/2$, appeared in [6]. The Airy constants are the special case $\ell = 1$. The case $\ell = 2$ corresponds to the recursion studied in [9, 10], which arises in the study of the Wiener index of Catalan trees. We have $C_1 = \frac{\Gamma(\ell-1/2)}{\sqrt{\pi}}$ and, for $n \geqslant 2$,

$$C_n \;=\; n\frac{\Gamma(n\ell+(n/2)-1)}{\Gamma((n-1)\ell+(n/2)-1)}C_{n-1}+\frac{1}{4}\sum_{k=1}^{n-1}\binom{n}{k}C_kC_{n-k}. \tag{11}$$

Define a_n by $C_n = n!\,g(n)a_n$ where $g(1) = 1$ and

$$g(m) \;=\; \prod_{k=2}^{m}\frac{\Gamma(k\ell+(k/2)-1)}{\Gamma((k-1)\ell+(k/2)-1)}.$$

Then (11) becomes

$$a_n \;=\; a_{n-1}+\sum_{k=1}^{n-1}\frac{g(k)g(n-k)}{4g(n)}a_ka_{n-k},$$

so $f(n,k) = g(k)g(n-k)/4g(n)$.

With a fixed and $x \to \infty$ and using 6.1.47 on p.257 of [1] (or using Stirling's formula), we have

$$\begin{aligned}\frac{\Gamma(x+a)}{\Gamma(x)} \;&=\; x^a\left(1+\frac{a(a-1)}{2x}+O(1/x^2)\right)\\ &=\; x^a\left(1+\frac{a-1}{2x}\right)^a\Big(1+O(1/x^2)\Big) &(12)\\ &=\; \left(x+\frac{a-1}{2}\right)^a\Big(1+O(1/x^2)\Big). &(13)\end{aligned}$$

When $m > 1$, (13) gives us

$$\begin{aligned}g(m) \;&=\; \prod_{k=2}^{m}\left(\frac{(2\ell+1)k-\ell-3}{2}\right)^{\ell}\Big(1+O(1/k^2)\Big)\\ &=\; \Theta(1)\left((\ell+1/2)^m\prod_{k=2}^{m}\left(k-\frac{\ell+3}{2\ell+1}\right)\right)^{\ell}\\ &=\; \Theta(1)\left((\ell+1/2)^m\,[a]_{m-1}\right)^{\ell}, \quad \text{where } a \;=\; \frac{3\ell-1}{2\ell+1}.\end{aligned}$$

Hence

$$f(n,k) \;=\; \Theta(1)\left|\frac{[a]_{k-1}\,[a]_{n-k-1}}{[a]_{n-1}}\right|^{\ell}.$$

where the absolute values have been introduced to allow for $a < 0$. A slight adjustment of the argument leading to (8) and (9) leads to

$$f(n,k) = O(n^{-\ell}(3k/2n)^{\ell(k-1)}) \quad \text{and} \quad \frac{f(n,k)}{f(n,1)} = O(n^{-\ell}(3k/2n)^{\ell(k-d-1)})$$

for $1 \leqslant k \leqslant n/2$. Hence Theorem 2 applies with $\alpha = \beta = \ell$, and a_n converges to a constant when $\ell > 1$ by Corollary 1. ∎

It is interesting to note that there is a simple relation between the sequence u_n in Example 1 and the sequence a_n in Example 3 with $\ell = 2$. It is not difficult to check that the $f(n,k)$ defined in Example 3 is exactly five times the $f(n,k)$ in Example 1: since $a_1 = 5u_1$, we have $a_n = 5u_n$ for all $n \geqslant 1$. This simple relation suggests a relationship between the number of maps on an orientable surface of genus g and the gth moment of a particular toll function on a certain type of trees. Using a bijective approach, Chapuy [4] recently found an expression for t_g as the gth moment of the labels in a random well-labelled tree.

3 A convolutional recursion arising from Painlevé I

The following is recursion (44) in [11].

$$\alpha_n = (n-1)^2\alpha_{n-1} + \sum_{k=2}^{n-2} \alpha_k \alpha_{n-k}, \ n \geqslant 1, \ n \geqslant 3. \tag{14}$$

It follows from Proposition 14 of [11] that, for $0 < \alpha_1 < 1$ and $\alpha_2 = \alpha_1 - \alpha_1^2$,

$$\alpha_n = c(\alpha_1)((n-1)!)^2 \left(1 - \frac{2\alpha_2(n-3)}{3(n-1)^2(n-2)^2} + \delta_n\right), \tag{15}$$

where $c(\alpha_1)$ depends only on α_1, and

$$\delta_n = O(1/n^4).$$

We note that α_n for $n \geqslant 3$ depends only on α_2. The proof of (15) relies on the fact that $0 < \alpha_2 < 1/4$ for $0 < \alpha_1 < 1$. It is conjectured in [11] that the asymptotic expression (15) actually holds for a wider range of values of α_1.

For $n \geqslant 1$, let

$$p_n = \frac{\alpha_n}{((n-1)!)^2}.$$

Then, as shown in [11], p_n satisfies recursion (1) with $d = 2$ and

$$f(n,k) = \left(\frac{(n-k-1)!(k-1)!}{(n-1)!}\right)^2.$$

We note here $f(n,2) = O(n^{-4})$. It follows from Theorem 2 that

$$p_n = p(1+\epsilon_n) \ \text{ for any } \alpha_2 > 0,$$

where $p = p(\alpha_2)$ is a positive constant and $\epsilon_n = O(1/n^3)$.

It is also interesting to note that, with $\alpha_1 = 1/50$, $\alpha_2 = 49/2500$, the sequence α_n is related to the sequence u_n in Example 1 by

$$\alpha_n = [1/5]_n \, [4/5]_{n-1} \, u_n.$$

As mentioned in [11], the formal series $v(t) = \sum_{n \geqslant 1} \alpha_n t^{-n}$ satisfies

$$t^2 v'' + tv' - (t + 2\alpha_1)v + tv^2 + \alpha_1 = 0, \tag{16}$$

and hence, with

$$t = \frac{8\sqrt{6}}{25} x^{5/2},$$

$y(x) = (x/6)^{1/2}(1 - 2v(t))$ satisfies the following Painlevé I:

$$y'' = 6y^2 - x.$$

This connection with Painlevé I is used in [8] to show that the sequence α_n is not holonomic (It follows that u_n and t_n in Example 1 are also not holonomic). The proof uses the fact that solutions to Painlevé I have infinitely many singularities and hence cannot satisfy a linear ODE with polynomial coefficients.

In the following we apply the techniques of [14] to prove that (15) holds for any complex constant α_1. It is convenient to introduce the formal series

$$u_0(z) = v(z^2) = \sum_{n=2}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} \alpha_n z^{-2n}.$$

It follows from (16) that $u = u_0(z)$ is a formal solution to the differential equation

$$\frac{1}{4}u'' + \frac{1}{4z}u' - \left(1 + \frac{2\alpha_1}{z^2}\right) u + u^2 + \frac{\alpha_1}{z^2} = 0.$$

The Stokes lines for this differential equation are the positive and the negative real axes. When the negative real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_1(z) = K e^{2z} z^{-1/2} \sum_{n=0}^{\infty} c_n z^{-n},$$

in which the Stokes multiplier K is a constant (depending on the constant α_1). To determine the coefficients c_n we observe that u_1 is a solution of the linear differential equation

$$\frac{1}{4}u_1'' + \frac{1}{4z}u_1' - \left(1 + \frac{2\alpha_1}{z^2} - 2u_0\right) u_1 = 0.$$

Hence, for the coefficients c_n we can take $c_0 = 1$ and for the others we have

$$nc_n = \frac{1}{4}\left(n - \frac{1}{2}\right)^2 c_{n-1} + 2\sum_{k=4}^{n+1} b_k c_{n+1-k}, \qquad n \geqslant 1.$$

The first five coefficients are

$$c_0 = 1, \quad c_1 = \frac{1}{16}, \quad c_2 = \frac{9}{512}, \quad c_3 = \frac{75}{8192} + \frac{2}{3}\alpha_2, \quad c_4 = \frac{3675}{524288} + \frac{13}{24}\alpha_2.$$

In a similar manner it can be shown that when the positive real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_2(z) = iKe^{-2z}z^{-1/2}\sum_{n=0}^{\infty}(-1)^n c_n z^{-n}.$$

This is all the information that is needed to conclude that

$$\alpha_n = b_{2n} \sim \frac{K}{\pi}\sum_{k=0}^{\infty}(-1)^k c_k \frac{\Gamma(2n-k-\frac{1}{2})}{2^{2n-k-(1/2)}}, \qquad \text{as} \ \ n \to \infty.$$

By taking the first 4 terms in this expansion we can verify that (15) holds for any complex constant α_1.

For more details see [12], [13] and [14]. (It's best to get the version of the first reference on the website `http://www.maths.ed.ac.uk/ adri/public.htm`.)

Acknowledgement We would like to thank Philippe Flajolet for bringing our attention to references [5] and [7]

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series - 55 (1964). Available online at `http://www.math.sfu.ca/∼cbm/aands/` and other sites.

[2] W. Balser, *From divergent series to analytic functions*, Springer-Verlag Lecture Notes, No 1582 (1994)

[3] E.A. Bender, Z.C. Gao and L.B. Richmond, The map asymptotics constant t_g, *Electron. J. Combin.* **15** (2008), R51.

[4] G. Chapuy, The structure of unicellular maps, and a connection between maps of positive genus and planar labelled trees, preprint, 2009.

[5] J. A. Fill , P. Flajolet, and N. Kapur, Singularity analysis, Hadamard products, and tree recurrences, *J. Comput. Appl. Math.* **174** (2005), 271–313.

[6] J. A. Fill and N. Kapur, Limiting distributions for additive functionals, *Theoret. Comput. Sci.* **326** (2004), 69–102.

[7] P. Flajolet and G. Louchard, Analytic variations on the Airy distribution, *Algorithmica* **31** (2001), 361–377.

[8] S. Garoufalidis, T. T. Lê, and Marcos Mariño , Analyicity of the free energy of a closed 3-manifold, preprint, 2008.

[9] S. Janson, The Wiener index of simply generated random trees, *Random Struct. Alg.* **22** (2003), 337–358.

[10] S. Janson and P. Chassaing, The center of mass of the ISE and the Wiener index of trees, *Elect. Comm. in Probab.* **9** (2004), 178-187.

[11] N. Joshi and A.V. Kitaev, On Boutroux's Tritronquée Solutions of the First Painleveé Equation, *Studies in Applied Math* **107** (2001), 253–291.

[12] Olde Daalhuis, A. B., Hyperasymptotic solutions of higher order linear differential equations with a singularity of rank one. *Proc. R. Royal Soc. Lond. A* 454, 1-29, (1998).

[13] Olde Daalhuis, A. B., Hyperasymptotics for nonlinear ODEs I: A Ricatti equation, *Proc. Royal Soc. Lond. A.* 461, 2503-2520, (2005).

[14] Olde Daalhuis, A. B., Hyperasymptotics for nonlinear ODEs II: The first Painlevé equation and a second-order Riccati equation, *Proc. Royal Soc. Lond. A.* 461, 3005-3021, (2005).